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SCHUR MULTIPLICATORS OF FINITE p-GROUPS WITH FIXED COCLASS

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ABSTRACT

We investigate the Schur multiplicators $M(G)$ of p-groups G using coclass theory. For $p > 2$ we show that there are at most finitely many p-groups G of coclass r with $|M(G)| \leq s$ for every r and s. We observe that this is not true for $p = 2$ by constructing infinite series of 2-groups G with coclass r and $|M(G)| = 1$. We investigate the Schur multiplicators of the 2-groups of coclass r further.

1. Introduction

Schur multiplicators of groups have been introduced by Schur [11] to study projective representations of groups. Since then, they have proved to be a powerful tool in group theory. For example, they play a role in Galois theory and they are relevant in the theory of central group extensions. We refer to [10] and [5] for an introduction to Schur multiplicators and for references on the topic.

The p-part of the Schur multiplicator of a finite group embeds into the Schur multiplicator of its Sylow p-subgroup. Hence the study of the Schur multiplicators of finite p-groups is of central interest in this area. The search for p-groups with trivial Schur multiplicator is of particular interest. See also Question 17 of [8].

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In this paper we investigate the Schur multiplicators of finite p-groups using the coclass as primary invariant: The **coclass** of a group G of order p^n and class c is $cc(G) = n - c$. The first main result of this paper is the following. (See Section 3 for a proof.)

THEOREM A: *Let $p > 2$ prime and $r \in \mathbb{N}$. For every $s \in \mathbb{N}$ there are only finitely many p-groups G of coclass r with $|M(G)| \leq s$.*

COROLLARY: *Let $p > 2$ prime and $r \in \mathbb{N}$. Then there are only finitely many p-groups of coclass r with trivial Schur multiplicator.*

It is easy to show that Theorem A and its Corollary are not valid for 2-groups. As an example, the groups

$$\langle g, t, c \mid g^{2^r} = c, t^{2^n} = c, c^2 = 1, t^g = t^{-1}, c^g = c^t = c \rangle$$

have order 2^{r+n+1}, coclass r and trivial Schur multiplicator. Hence, for every possible coclass r there are infinitely many 2-groups of coclass r with trivial Schur multiplicator.

Coclass theory provides a powerful tool to study the Schur multiplicators of the finite p-groups of coclass r further. For this purpose we use the graph, $\mathcal{G}(p, r)$; its vertices correspond to the isomorphism types of the finite p-groups of coclass r and there is an edge between G and H if $G \cong H/N$ where N is the last non-trivial term of the lower central series of H. Every infinite pro-p-group S of coclass r defines a maximal coclass tree $\mathcal{T}(S)$ in $\mathcal{G}(p, r)$: this is the infinite subtree of $\mathcal{G}(p, r)$ consisting of all descendants of $S/\gamma_i(S)$, where $\gamma_i(S)$ is the ith term of the lower central series of S and i is minimal such that $cc(S/\gamma_i(S)) = r$ and $S/\gamma_i(S)$ is not isomorphic to a quotient of an infinite pro-p-group $R \not\cong S$ with $cc(R) = r$. By Theorem D of the coclass theorems, see [6], there are only finitely many isomorphism types of infinite pro-p-groups of coclass r. Thus $\mathcal{G}(p, r)$ consists of finitely many maximal coclass trees and finitely many other groups. A second result of this paper is the following. (See Section 4 for a proof.)

THEOREM B: *Let S be an infinite pro-2-group of coclass r and let $\mathcal{T} = \mathcal{T}(S)$.*

(a) *If $|M(S)| = \infty$, then for every s there are only finitely many $G \in \mathcal{T}$ with $|M(G)| \leq s$.*
(b) *If $|M(S)| < \infty$, then there is an $s = s(S) \in \mathbb{N}$ with $|M(G)| \leq s$ for all groups $G \in \mathcal{T}$.*

(c) *If* $|M(S)| \neq 1$, *then there are only finitely many groups* $G \in \mathcal{T}$ *with* $|M(G)| = 1$.

It follows from [7] that the ranks of the Schur multiplicators of the p-groups of coclass r are bounded above by a function in p and r. In the course of our proof for Theorem B, we exhibit upper and lower bounds for the ranks and the exponents of the Schur multiplicators of the p-groups in a tree $\mathcal{T}(S)$. (See Section 4.)

Theorem B implies that almost all of the infinitely many 2-groups of coclass r with trivial Schur multiplicator are contained in trees $\mathcal{T}(S)$ for infinite pro-2-groups S with coclass r and $|M(S)| = 1$. In [2] it has been observed that such groups, S, exist for every r. In Section 5 we demonstrate how Theorem B can be used to determine infinite sequences of 2-groups with trivial Schur multiplicator.

We consider the results of [1] and [3] to investigate the situation further: these assert that the 2-groups of coclass r in a tree $\mathcal{T}(S)$ fall into finitely many 'periodicity classes' and finitely many 'sporadic groups' (see also Section 2 for background). Such periodicity classes also exist for p-groups with $p > 2$, but they may not contain almost all p-groups of coclass r in a tree $\mathcal{T}(S)$ in this case.

Using the computer algebra system Gap [12], we determined the Schur multiplicators of a large collection of 2-groups of coclass at most 3 and of 3-groups of coclass at most 2. These experiments suggest that the periodicity classes of p-groups have a major influence on the structure of the Schur multiplicators of the p-groups of coclass r. The following conjecture describes the results of our experiments.

Conjecture: *Let* $(G_0, G_1, \ldots)$ *be a periodicity class of* p*-groups. There exist* $f, t \in \mathbb{N}_0$ *and* $l_j, m_j \in \mathbb{N}_0$ *for* $1 \leq j \leq t$ *such that for every* $i \geq f$ *it follows that*

$$M(G_i) \cong C_{p^{m_1+il_1}} \times \cdots \times C_{p^{m_t+il_t}}.$$

If this conjecture is true, then the infinitely many 2-groups of coclass r with trivial Schur multiplicator fall into finitely many periodicity classes and finitely many other groups. Theorem B asserts that a periodicity class of 2-groups with trivial Schur multiplicator can only arise in a tree $\mathcal{T}(S)$ where S has trivial Schur multiplicator.

2. Notation and preliminaries

In this section we summarise some background on coclass theory of finite p-groups since we need it later. We refer to [6] for details and information on the state of the art of coclass theory.

In general, we denote with $G = \gamma_1(G) > \gamma_2(G) > \cdots$ the lower central series of a group G and we write $G_i = G/\gamma_i(G)$ for its quotients. The coclass of an infinite pro-p-group S is then defined as $cc(S) = \lim_{i\to\infty} cc(S_i)$.

2.1. Infinite pro-p-groups of finite coclass and their trees. An infinite pro-p-group S of coclass r has the structure of a **uniserial p-adic pre-space group**; that is, there exist l and d such that $\gamma_i(S) \cong \mathbb{Z}_p^d$ and $[\gamma_i(S) : \gamma_{i+1}(S)] = p$, for every $i \geq l$. The integer d is called the **dimension** of S.

Its maximal coclass tree $\mathcal{T}(S)$ contains exactly one maximal infinite path $S_i, S_{i+1}, \ldots$ starting at the root S_i of the tree $\mathcal{T}(S)$. This maximal infinite path is called the **main line** of $\mathcal{T}(S)$.

For every $j \geq i$ we define the subtree $\mathcal{B}_j(S)$ of $\mathcal{T}(S)$ as the subgraph containing all descendants of S_j which are not descendants of S_{j+1}. By construction, every $\mathcal{B}_j(S)$ is a finite subtree of $\mathcal{T}(S)$ and it is called a **branch**.

The subtree $\mathcal{T}(S,k)$ contains all groups with distanced $\leq k$ from the main line and is called a **shaved tree**. Its branches are denoted with $\mathcal{B}_j(S,k)$.

2.2. Periodicity classes. Let S be an infinite pro-p-group of finite coclass and dimension d. It has been conjectured in [9] (Conjecture P) and proved in [1] and [3] that for every $k \in \mathbb{N}$ there exists an $f = f(k,S)$, such that the branches $\mathcal{B}_j(S,k)$ with $j \geq f$ of the shaved tree $\mathcal{T}(S,k)$ satisfy a periodic pattern; that is, there exists a graph isomorphism π with

$$\pi : \mathcal{B}_j(S,k) \to \mathcal{B}_{j+d}(S,k) \quad \text{for every } j \geq f.$$

In [3] there is an explicit group theoretic construction outlined which underpins the isomorphism π and allows one to construct the image of a group G under π. For $G \in \mathcal{B}_j(S,k)$ we denote with $(G, \pi(G), \pi^2(G), \ldots)$ its **periodicity class**.

1 Theorem (See [3]): *Let S be an infinite pro-p-group of finite coclass and dimension d and let $(G, \pi(G), \ldots)$ be a periodicity class in $\mathcal{T}(S)$. Suppose that $G \in \mathcal{B}_j(S)$ has distance e to the main line. Then every $\pi^i(G)$ has distance e to the main line and is an extension of $\gamma_{j+id}(G) \cong \gamma_{j+id}(S)/\gamma_{j+id+e}(S)$ by S_{j+id}.*

If $p = 2$, then there exists an integer $k = k(S)$ such that $\mathcal{T}(S) = \mathcal{T}(S, k)$, see [6], Theorem 11.3.7. It follows that every tree $\mathcal{T}(S)$ and thus also all of $\mathcal{G}(2, r)$ consists of finitely many periodicity classes and finitely many 'sporadic groups'; that is, groups not contained in a periodicity class.

3. Uniserial extensions

An extension G of N by Q is called **uniserial** if $N \leq G'$ and the series defined by $N_0 := N$ and $N_{i+1} = [N_i, G]$ satisfies $[N_i : N_{i+1}] = p$ for some prime p.

In this section we investigate first the Schur multiplicators of uniserial extensions. Then we apply these results to the groups in a maximal coclass $\mathcal{T}(S)$ using that every group G in a branch $\mathcal{B}_j(S)$ of $\mathcal{T}(S)$ is a uniserial extension of $\gamma_j(G)$ by the main-line group S_j.

Theorem A and its corollary as well as Theorem B (a) and (c) follow as application of the results of this section.

3.1. Maps induced by projection. Let G be an extension of N by Q. The 5-term homology sequence, see [10], 11.4.17, induces the exact sequence

$$M(G) \xrightarrow{\beta_{G,Q}} M(Q) \xrightarrow{\gamma_{G,Q}} N/[N,G] \xrightarrow{\delta_{G,Q}} G/G'.$$

To give an explicit description for the maps in this sequence, let F/R be a presentation for G and let F/U be a presentation for Q such that $R \leq U$. By Hopf's formula we can identify $M(G) = (R \cap F')/[R, F]$ and $M(Q) = (U \cap F')/[U, F]$. Further, it follows that $N \cong U/R$ and $G/G' \cong F/RF'$. We denote with U_i/R the preimage of N_i in U/R for $i \in \mathbb{N}_0$. The maps in the above exact sequence are then defined by

$$\beta_{G,Q} : (R \cap F')/[R, F] \to (U \cap F')/[U, F] : r[R, F] \mapsto r[U, F];$$
$$\gamma_{G,Q} : (U \cap F')/[U, F] \to U/U_1 : u[U, F] \mapsto uU_1;$$
$$\delta_{G,Q} : U/U_1 \to G/G' : uU_1 \mapsto uRF'.$$

We consider the exact sequence for uniserial extensions using this notation.

2 Lemma: *Let G be a non-trivial uniserial extension of N by Q. Then*

a) $\delta_{G,Q} = 0$ *and* $\gamma_{G,Q}$ *is surjective with* $\mathrm{Im}(\gamma_{G,Q}) \cong C_p$.
b) $\mathrm{Im}(\beta_{G,Q}) = \mathrm{Ker}(\gamma_{G,Q}) = (U_1 \cap F)/[U, F]$.

In particular, we obtain the exact sequence $M(G) \to M(Q) \to C_p \to \{1\}$.

Proof. (a) As the extension is uniserial, it follows that $N \leq G'$ and hence $U \leq RF'$. Thus $\delta_{G,Q} = 0$ and $\gamma_{G,Q}$ is surjective. As $U/U_1 \cong N/N_1 \cong C_p$, the result follows.

(b) $\mathrm{Ker}(\gamma_{G,Q}) = (U \cap F' \cap U_1)/[U,F] = (U_1 \cap F')/[U,F]$. ∎

In the special case that G is a central extension of $N \cong C_p$ by Q, we can also describe the kernel of $\beta_{G,Q}$ to some extent.

3 Lemma: *Let G be a uniserial extension of $N \cong C_p$ by Q. Then* $\mathrm{Ker}(\beta_{G,Q})$ *is elementary abelian of rank at most* $\mathrm{rk}(G/G')$.

Proof. By [4], the 5-term homology sequence can be completed to the exact sequence $N \otimes (G/G') \to M(G) \to M(Q) \to C_p \to \{1\}$ in the case of a central extension. The left most map in this extended sequence is induced by $U/R \times F/RF' \to (R \cap F')/[R,F] : (xR, yRF') \mapsto [x,y][R,F]$. As $N \otimes (G/G')$ is elementary abelian of rank at most $\mathrm{rk}(G/G')$, the result follows. ∎

3.2. Applications to maximal coclass trees. Let S be an infinite pro-p-group of coclass r. Then S is a uniserial extension of $\gamma_j(S)$ by S_j for every large enough j. We define

$$I_j = \mathrm{Im}(\beta_{S,S_j}) \quad \text{and} \quad K_j = \mathrm{Ker}(\beta_{S,S_j}).$$

4 Theorem: *Let G be a group in a branch $\mathcal{B}_j(S)$ of the maximal coclass tree $\mathcal{T}(S)$. Then $|I_j|$ divides $|M(G)|$.*

Proof. The group G is an extension of $\gamma_j(G)$ by S_j. Lemma 2 yields that I_j and $\mathrm{Im}(\beta_{G,S_j})$ are both subgroups of index p in the finite abelian p-group $M(S_j)$. Thus $|I_j| = |\mathrm{Im}(\beta_{G,S_j})|$ divides $|M(G)|$. ∎

We investigate $M(S_j)$ and I_j further. For this purpose we consider presentations $F/R_j = S_j$ and $F/R = S$ such that $R \leq \cdots \leq R_{j+1} \leq R_j \leq \cdots$. Then we obtain the following explicit descriptions of I_j and K_j.

5 Lemma: $I_j = (R_{j+1} \cap F')/[R_j, F]$ *and* $K_j = (R \cap [R_j, F])/[R,F]$ *for every* $j \in \mathbb{N}$.

Proof. By construction: $I_j = ([R_j,F]R \cap F')/[R_j,F] = (R_{j+1} \cap F')/[R_j,F]$ and $K_j = (R \cap F' \cap [R_j,F])/[R,F] = (R \cap [R_j,F])/[R,F]$. ∎

6 Lemma: $\bigcap_{j \in \mathbb{N}} K_j = \{1\}$.

Proof. $\bigcap_{j\in\mathbb{N}}(R\cap[R_j,F]) = R\cap[(\cap_{j\in\mathbb{N}}R_j),F] = R\cap[R,F] = [R,F]$. ∎

The next theorem provides the main basis for our proofs of Theorem A with its corollary and of Theorem B (a) and (c).

7 Theorem:

(a) *If $M(S)$ is finite, then there exists an $i\in\mathbb{N}$ such that $I_j = M(S)$ for all $j\geq i$.*
(b) *If $M(S)$ is infinite, then the orders of I_j are unbounded.*

Proof. (a) By Lemma 6 there exists an $i\in\mathbb{N}$ such that $K_j = K_i = \{1\}$ for all $j\geq i$. This yields that $I_j\cong M(S)/K_j = M(S)$ for all $j\geq i$.
(b) We note that I_j is finite with $I_j\cong M(S)/K_j$. Thus Lemma 6 yields that the orders of I_j are unbounded. ∎

Theorems 7 and 4 have the following immediate corollaries.

8 Corollary:

(a) *If $M(S)$ is non-trivial, then there exists an l such that $|M(G)|\neq 1$ for every group G in a branch $\mathcal{B}_j(S)$ with $j\geq l$.*
(b) *If $M(S)$ is infinite, then for every $s\in\mathbb{N}$ there exists an $l = l(s)$ such that $|M(G)|\geq s$ for every G in a branch $\mathcal{B}_j(S)$ with $j\geq l$.*

Corollary 8 implies Theorem B (a) and (c) directly, as branches are finite subtrees of $\mathcal{T}(S)$. Further, the results of [2] assert that $M(S)$ is infinite if $p>2$. As $\mathcal{G}(p,r)$ contains only finitely many maximal coclass trees and finitely many groups outside, Theorem A and its corollary also follow from Corollary 8.

4. Bounds for rank and exponent

In this section we investigate the Schur multiplicators of the groups in a maximal coclass tree $\mathcal{T}(S)$ further, and we introduce bounds on rank and exponent for them. Theorem B (b) will follow as application of the results of this section.

As a first step, we describe the Schur multiplicators of the main line groups in $\mathcal{T}(S)$ more precisely. We continue to use the notation of Section 3.

9 Theorem: *$M(S_i) = I_i\times C_p$ for every large enough i.*

Proof. By Lemma 2, it follows that $[M(S_i):I_i] = p$, provided that i is large enough so that S is a uniserial extension of $\gamma_i(S)$ by S_i. Note that $\beta_{S,S_{i-1}} =$

$\beta_{S,S_i}\beta_{S_i,S_{i-1}}$. Thus $K = \mathrm{Ker}(\beta_{S_i,S_{i-1}})$ supplements I_i in $M(S_i)$. Lemma 3 yields that K is elementary abelian. Hence it contains a subgroup of order p complementing I_i in $M(S_i)$. ∎

Theorem 9 implies that $M(S_i)$ is cyclic of order p for every i if $M(S)$ is trivial. Further, it induces the following bounds on rank and exponent of the Schur multiplicators of the main-line groups. Recall that the Schur multiplicator of an infinite pro-p-groups of finite coclass has finite rank.

10 THEOREM: *Let S be an infinite pro-p-group of coclass r with $|M(S)| > 1$. Then for every large enough $i \in \mathbb{N}$ it follows that*

a) $\mathrm{rk}(M(S_i)) = \mathrm{rk}(M(S)) + 1$.
b) *If $M(S)$ is finite, then* $\exp(M(S_i)) = \exp(M(S))$.
c) *If $M(S)$ is infinite, then* $\exp(M(S_i))$ *is unbounded with* $\exp(M(S_i)) \leq p^{\frac{1}{2}(i+r)}$.

Proof. a), b) and the first part of c) follow directly from Theorem 9 and 7. It remains to prove the upper bound on $\exp(M(S_i))$ if $M(S)$ is infinite. A theorem of Schur [11] asserts that $\exp(M(G))^2 \mid |G|$ for a finite group G. This yields the desired upper bound, as $|S_i| = p^{i+r}$. ∎

These bounds on rank and exponent of the Schur multiplicators of the main-line groups induce similar bounds on the ranks and exponents of the Schur multiplicators of the groups in $\mathcal{T}(S,k)$ for fixed k.

11 THEOREM: *Let S be an infinite pro-p-group of coclass r and denote $f = \mathrm{rk}(S/S')$ and $b = \mathrm{rk}(M(S))$. Let G be a group in $\mathcal{B}_j(S,k)$ for some arbitrary $k \in \mathbb{N}$ and some large enough $j \in \mathbb{N}$. Denote $c_j = \exp(M(S_j))$. Then*

$$b \leq \mathrm{rk}(M(G)) \leq b + kf + 1 \quad \text{and} \quad c_j/p \leq \exp(M(G)) \leq p^k c_j.$$

Proof. Let $J = \mathrm{Im}(\beta_{G,S_j})$. By Lemma 2, J is a subgroup of index p in $M(S_j)$. We first consider the rank of $M(G)$ and find that $\mathrm{rk}(M(S)) + 1 = \mathrm{rk}(M(S_j) \geq \mathrm{rk}(J) \geq \mathrm{rk}(M(S_j)) - 1 = \mathrm{rk}(M(S))$ by Theorem 10. As G has distance at most k from the main line, it is an at most k-fold iterated central extension of C_p and S_j. By Lemma 3, the kernel of each of the involved projection maps is elementary abelian of rank at most f. Thus $|M(G)| \leq |J|p^{kf}$. Hence $\mathrm{rk}(M(G)) \leq \mathrm{rk}(J) + kf \leq \mathrm{rk}(M(S)) + 1 + kf$. A similar argument yields the inequality for the exponent. ∎

For $p = 2$ there exists a $k = k(r)$ such that $\mathcal{B}_j(S, k) = \mathcal{B}_j(S)$ for every j and S. Thus Theorem B (b) follows directly from Theorems 10 and 11. Further, we obtain another proof that the ranks of the Schur multiplicators of the 2-groups of coclass r are bounded by a function in r; a result that also follows from [7].

5. Examples of 2-groups with trivial Schur multiplicator

Theorem B provides a recipe for finding infinite series of 2-groups with given coclass r and trivial Schur multiplicator: first we determine the infinite pro-2-groups S of coclass r with trivial Schur multiplicator and then we investigate the groups in $\mathcal{T}(S)$. If the Conjecture of Section 1 is true, then almost all of the groups in $\mathcal{T}(S)$ with trivial Schur multiplicator fall into periodicity classes. Our experimental evidence supports this and in this section we report on some of our experiments.

The infinite pro-2-groups of coclass $r \in \{1, 2, 3\}$ have been determined in [9]: There is 1 group of coclass 1, there are 5 groups of coclass 2 and there are 54 groups of coclass 3. Among these there are 5 groups with trivial Schur multiplicator as shown in [2]. We recall these groups here for completeness.

$$S_1 = \langle a, t \mid a^2 = 1, t^a = t^{-1} \rangle \qquad (cc1)$$

$$S_2 = \langle a, t \mid a^4 = 1, t^a = t^{-1} \rangle \qquad (cc2)$$

$$S_3 = \langle a, t \mid a^8 = 1, t^a = t^{-1} \rangle \qquad (cc3)$$

$$S_4 = \langle a, t, b \mid a^2 = b^2, b^4 = 1, t^a = t^{-1}b, b^a = b^{-1}, b^t = b^{-1} \rangle \qquad (cc3)$$

$$S_5 = \langle a, b, c, t_1, t_2, d \mid a^2 = d, b^2 = t_2, c^2 = t_1, d^2 = 1, b^a = c, c^a = b, \qquad (cc3)$$
$$c^b = ct_1^{-1}t_2d, t_1^a = t_2, t_2^a = t_1, t_1^b = t_1^{-1}, t_2^c = t_2^{-1} \rangle$$

The tree $\mathcal{T}(S_1)$ is the (unique) maximal coclass tree of $\mathcal{G}(2, 1)$. This tree is well-understood. It contains two infinite series of 2-groups with trivial Schur multiplicator: the quaternion groups and the semi-dihedral groups. Both of these series form a periodicity class in $\mathcal{G}(2, 1)$. The third periodicity class in $\mathcal{G}(2, 1)$ contains only dihedral groups and the Schur multiplicators of the groups in this class are all cyclic of order 2.

We determined large (but finite) parts of $\mathcal{T}(S)$ for various pro-2-groups S of coclass at most 3 including the groups $S_1, \ldots, S_5$ and we computed the Schur multiplicators of the groups obtained in $\mathcal{T}(S)$. These experiments provide strong support for the Conjecture of Section 1. In particular, they support the conjecture that the Schur multiplicators of the groups in a single periodicity class of $\mathcal{T}(S_i)$ are all isomorphic if $M(S_i)$ is finite. The following tables exhibits the number of periodicity classes and the conjectured number of classes with trivial Schur multiplicator for $1 \leq i \leq 5$:

	S_1	S_2	S_3	S_4	S_5
total number of periodicity classes	3	4	6	5	18
number of classes with trivial multiplicator	2	2	3	3	10

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